# Upper bounds of the bichromatic number of some graphs

Yueping Shi

School of Mathematics,

Sun Yat-sen University

Joint work with Baogang Xu

- Basic definitions

- Basic definitions
- Known results

- Basic definitions
- Known results
- Our results

- Basic definitions
- Known results
- Our results
- The proof

- Basic definitions
- Known results
- Our results
- The proof
- Open problems

# 1. Basic definitions

# 1. Basic definitions

- $(k, l)$-**coloring**: a $(k, l)$-coloring of a graph $G$ is a partition of the vertex set of $G$ into $k + l$ (possibly empty) subsets

$$S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$$

such that each $S_i$ is an independent set and each $C_j$ is a clique in $G$.

# 1. Basic definitions

- $(k, l)$-**coloring**: a $(k, l)$-coloring of a graph $G$ is a partition of the vertex set of $G$ into $k + l$ (possibly empty) subsets

$$S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$$

such that each $S_i$ is an independent set and each $C_j$ is a clique in $G$.

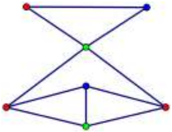
- Call a graph $G$ is $(k, l)$-**colorable** if $G$ has a $(k, l)$-coloring.

# 1. Basic definitions

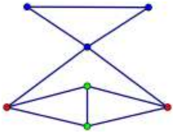
- $(k, l)$-**coloring**: a $(k, l)$-coloring of a graph $G$ is a partition of the vertex set of $G$ into $k + l$ (possibly empty) subsets

$$S_1, S_2, \ldots, S_k, C_1, C_2, \ldots, C_l$$

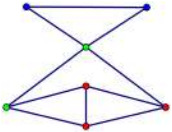
  such that each $S_i$ is an independent set and each $C_j$ is a clique in $G$.
- Call a graph $G$ is $(k, l)$-**colorable** if $G$ has a $(k, l)$-coloring.
- **Example**.



$(3, 0)$-colorable



$(1, 2)$-colorable



$(0, 3)$-colorable

# 1. Basic definitions

# 1. Basic definitions

Let $\overline{G}$ denote the **complement** of a graph $G$.

# 1. Basic definitions

Let $\overline{G}$ denote the **complement** of a graph $G$.

The **clique covering number** $\theta(G)$: the minimum number of cliques needed to cover the vertex set of a graph $G$.

# 1. Basic definitions

Let $\overline{G}$ denote the **complement** of a graph $G$.

The **clique covering number** $\theta(G)$: the minimum number of cliques needed to cover the vertex set of a graph $G$.

- $G$ is $(k, l)$-colorable iff $\overline{G}$ is $(l, k)$-colorable.

# 1. Basic definitions

Let $\overline{G}$ denote the **complement** of a graph $G$.

The **clique covering number** $\theta(G)$: the minimum number of cliques needed to cover the vertex set of a graph $G$.

- $G$ is $(k, l)$-colorable iff $\overline{G}$ is $(l, k)$-colorable.
- $(k, 0)$-colorable graphs are precisely $k$-colorable graphs.

# 1. Basic definitions

Let $\overline{G}$ denote the **complement** of a graph $G$.

The **clique covering number** $\theta(G)$: the minimum number of cliques needed to cover the vertex set of a graph $G$.

- $G$ is $(k, l)$-colorable iff $\overline{G}$ is $(l, k)$-colorable.
- $(k, 0)$-colorable graphs are precisely $k$-colorable graphs.
- $(0, l)$-colorable graphs are exactly those graphs of clique covering number at most $l$.

# 1. Basic definitions

# 1. Basic definitions

- The **bichromatic number** of $G$:

  $\chi^b(G) = \min\{r : \forall k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$

  [Prömel and Steger (1993)]

# 1. Basic definitions

- The **bichromatic number** of $G$:

  $\chi^b(G) = \min\ \{r : \forall k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$

  [Prömel and Steger (1993)]

- Remark: $\chi^b(G) = \chi^b(\overline{G})$.

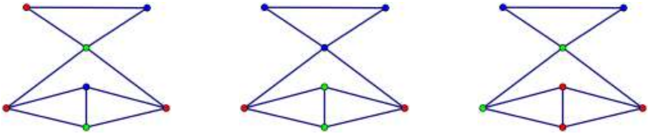
# 1. Basic definitions

- The **bichromatic number** of $G$:

  $\chi^b(G) = \min\{r : \forall k, l \text{ with } k+l=r,\ G \text{ is } (k,l)\text{-colorable}\}$

  [Prömel and Steger (1993)]

- Remark: $\chi^b(G) = \chi^b(\overline{G})$.
- **An example for the case** $\chi^b(G) = 4$. Not $(2,1)$-colorable!



$(3,0)$-colorable $(1,2)$-colorable $(0,3)$-colorable

$\Rightarrow$ $(4,0)$-colorable,$(3,1)$-colorable,$(2,2)$-colorable,$(1,3)$-colorable,$(0,4)$-colorable.

# 1. Basic definitions

- The **cochromatic number** of $G$:

  $\chi^c(G) = \min \{r : \exists k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$

  [Lesniak and Straight (1977)]

# 1. Basic definitions

- The **cochromatic number** of $G$:

  $\chi^c(G) = \min\ \{r : \exists k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$

  [Lesniak and Straight (1977)]

- Let $\chi(G)$ and $\theta(G)$ denote the **chromatic number** and the **clique covering number** of $G$, respectively. $(\theta(G) = \chi(\overline{G}))$

# 1. Basic definitions

- The **cochromatic number** of $G$:

  $\chi^c(G) = \min \{r : \exists k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$

  [Lesniak and Straight (1977)]

- Let $\chi(G)$ and $\theta(G)$ denote the **chromatic number** and the **clique covering number** of $G$, respectively. $(\theta(G) = \chi(\overline{G}))$

- Upper bound:

$$\chi^c(G) \leq \min \{\chi(G), \theta(G)\}$$

[Lesniak and Straight (1977)]

# 1. Basic definitions

$$\chi^b(G) = \min \{r : \forall k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$$

# 1. Basic definitions

$$\chi^b(G) = \min \{r : \forall k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$$

- Lower bound:

$$\chi^b(G) \geq \max \{\chi(G), \theta(G)\}.$$

[Prömel and Steger (1993)]

# 1. Basic definitions

$$\chi^b(G) = \min\ \{r : \forall k, l \text{ with } k + l = r,\ G \text{ is } (k, l)\text{-colorable}\}$$

- Lower bound:

$$\chi^b(G) \geq \max\ \{\chi(G), \theta(G)\}.$$

[Prömel and Steger (1993)]

- Upper bound:

$$\chi^b(G) \leq \chi(G) + \theta(G) - 1.$$

[Prömel and Steger (1993)]

**Proof:** If $k + l = \chi(G) + \theta(G) - 1$, then $k \geq \chi(G)$ or $l \geq \theta(G)$.

It follows that $G$ is $(k, l)$-colorable. □

# 1. Basic definitions

- **Complete $n$-partite graph**: a $n$-partite graph (i.e., a set of graph vertices admits a partition into $n$ classes s.t. no two vertices within the same class are adjacent) s.t. every pair of vertices from different classes are adjacent.

# 1. Basic definitions

- **Complete $n$-partite graph**: a $n$-partite graph (i.e., a set of graph vertices admits a partition into $n$ classes s.t. no two vertices within the same class are adjacent) s.t. every pair of vertices from different classes are adjacent.
- Let $K_{p_1,p_2,\ldots,p_n}$ be the **complete $n$-partite graph** with $p_i$ vertices in the $i$-th partite set, $1 \leq i \leq n$, $p_0 = 0 < p_1 \leq p_2 \leq \ldots \leq p_n$.

# 1. Basic definitions

- **Complete $n$-partite graph**: a $n$-partite graph (i.e., a set of graph vertices admits a partition into $n$ classes s.t. no two vertices within the same class are adjacent) s.t. every pair of vertices from different classes are adjacent.
- Let $K_{p_1,p_2,\ldots,p_n}$ be the **complete $n$-partite graph** with $p_i$ vertices in the $i$-th partite set, $1 \leq i \leq n$, $p_0 = 0 < p_1 \leq p_2 \leq \ldots \leq p_n$.

**Proposition (Lesniak and Straight, Ars Combin., 1977)**

$$\chi^c(K_{p_1,p_2,\ldots,p_n}) = min\,\{n - i + p_i \mid 0 \leq i \leq n\}.$$

# 1. Basic definitions

- **Complete** $n$**-partite graph**: a $n$-partite graph (i.e., a set of graph vertices admits a partition into $n$ classes s.t. no two vertices within the same class are adjacent) s.t. every pair of vertices from different classes are adjacent.
- Let $K_{p_1,p_2,\ldots,p_n}$ be the **complete** $n$**-partite graph** with $p_i$ vertices in the $i$-th partite set, $1 \leq i \leq n$, $p_0 = 0 < p_1 \leq p_2 \leq \ldots \leq p_n$.

**Proposition (Lesniak and Straight, Ars Combin., 1977)**

$$\chi^c(K_{p_1,p_2,\ldots,p_n}) = min\,\{n - i + p_i \mid 0 \leq i \leq n\}.$$

**Proposition**

$$\chi^b(K_{p_1,p_2,\ldots,p_n}) = max\,\{n - i + p_i \mid 0 \leq i \leq n\}.$$

**Proposition**

$$\chi^b(K_{p_1,p_2,\ldots,p_n}) = max\ \{n - i + p_i \mid 0 \le i \le n\}.$$

**Proof:** Let $G = K_{p_1,p_2,\ldots,p_n}$. Consider $\overline{G} = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$.

Let $n - k + p_k = \max\ \{n - i + p_i \mid 0 \le i \le n\}$.

# 1. Basic definitions

**Proposition**

$$\chi^b(K_{p_1,p_2,\ldots,p_n}) = max\ \{n - i + p_i \mid 0 \leq i \leq n\}.$$

**Proof:** Let $G = K_{p_1,p_2,\ldots,p_n}$. Consider $\overline{G} = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$.

Let $n - k + p_k = \max \{n - i + p_i \mid 0 \leq i \leq n\}$.

- $\chi^b(\overline{G}) \leq n - k + p_k$.

  (Since $\overline{G}$ is $(p_i, n - i)$-colorable, $\overline{G}$ is $(n - k + p_k - (n - i), n - i)$-colorable.)

## Proposition

$$\chi^b(K_{p_1,p_2,\ldots,p_n}) = max\ \{n - i + p_i \mid 0 \le i \le n\}.$$

**Proof:** Let $G = K_{p_1,p_2,\ldots,p_n}$. Consider $\overline{G} = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$.

Let $n - k + p_k = \max\ \{n - i + p_i \mid 0 \le i \le n\}$.

- $\chi^b(\overline{G}) \le n - k + p_k$.
  (Since $\overline{G}$ is $(p_i, n - i)$-colorable, $\overline{G}$ is $(n - k + p_k - (n - i), n - i)$-colorable.)
- $\chi^b(\overline{G}) > n - k + p_k - 1$.
  (If $1 \le k \le n$, then $\overline{G}$ is not $(p_k - 1, n - k)$-colorable.
  If $k = 0$, then $\overline{G}$ is not $(0, n - 1)$-colorable.)

**Proposition**

$$\chi^b(K_{p_1,p_2,\ldots,p_n}) = max\ \{n - i + p_i \mid 0 \leq i \leq n\}.$$

**Proof:** Let $G = K_{p_1,p_2,\ldots,p_n}$. Consider $\overline{G} = K_{p_1} \cup K_{p_2} \cup \ldots \cup K_{p_n}$.

Let $n - k + p_k = \max\ \{n - i + p_i \mid 0 \leq i \leq n\}$.

- $\chi^b(\overline{G}) \leq n - k + p_k$.

  (Since $\overline{G}$ is $(p_i, n - i)$-colorable, $\overline{G}$ is $(n - k + p_k - (n - i), n - i)$-colorable.)

- $\chi^b(\overline{G}) > n - k + p_k - 1$.

  (If $1 \leq k \leq n$, then $\overline{G}$ is not $(p_k - 1, n - k)$-colorable.

  If $k = 0$, then $\overline{G}$ is not $(0, n - 1)$-colorable.)

- $\chi^b(G) = \chi^b(\overline{G}) = \max\ \{n - i + p_i \mid 0 \leq i \leq n\}$.

# 2. Known results

The **independence number** $\alpha(G)$: the size of a maximum independent vertex set of a graph $G$.

# 2. Known results

The **independence number** $\alpha(G)$: the size of a maximum independent vertex set of a graph $G$.

**Proposition (Epple and Huang, JGT, 2010)**

*For every graph $G$ on $n$ vertices, $\chi^b(G) \geq \sqrt{n}$.*

# 2. Known results

The **independence number** $\alpha(G)$: the size of a maximum independent vertex set of a graph $G$.

**Proposition (Epple and Huang, JGT, 2010)**

*For every graph $G$ on $n$ vertices, $\chi^b(G) \geq \sqrt{n}$.*

- **Proof:** $\chi^b(G) \cdot \chi^b(G) \geq \chi(G) \cdot \theta(G) \geq \chi(G) \cdot \alpha(G) \geq n$. □

The **independence number** $\alpha(G)$: the size of a maximum independent vertex set of a graph $G$.

**Proposition (Epple and Huang, JGT, 2010)**

*For every graph $G$ on $n$ vertices, $\chi^b(G) \geq \sqrt{n}$.*

- **Proof:** $\chi^b(G) \cdot \chi^b(G) \geq \chi(G) \cdot \theta(G) \geq \chi(G) \cdot \alpha(G) \geq n$. □

**Theorem (Epple and Huang, JGT, 2010)**

*The problem of computing the bichromatic number of a graph is NP-hard.*

# 2. Known results

The **bidegree** of $G$: $\Delta^b(G) = \max\{\Delta(G), \Delta(\overline{G})\}$.

## 2. Known results

The **bidegree** of $G$: $\Delta^b(G) = \max\{\Delta(G), \Delta(\overline{G})\}$.

- Brooks-type theorem:

# 2. Known results

The **bidegree** of $G$: $\Delta^b(G) = \max\{\Delta(G), \Delta(\overline{G})\}$.

- Brooks-type theorem:

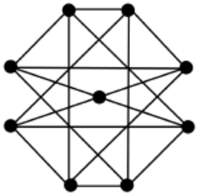
**Theorem (Epple and Huang, JGT, 2014)**

*For any graph $G$,*

$$\chi^b(G) \leq \Delta^b(G) + 1.$$

*Equality holds iff $G$ is one of $K_n, K_{m,m}, C_5, Q$ or their complements.*

- The graph $Q$ in the above theorem is depicted below.



# 2. Known results

**cograph**: a graph not contain $P_4$ (i.e., the path with four vertices) as an induced subgraph.

[Corneil, Lerchs and Burlingham (1981)]

# 2. Known results

**cograph**: a graph not contain $P_4$ (i.e., the path with four vertices) as an induced subgraph.

[Corneil, Lerchs and Burlingham (1981)]

**box cograph**: a box cograph is a cograph $G$ having exactly $\chi(G)\theta(G)$ vertices.

The class of box cographs $G$ is denoted by $\mathcal{B}(r, s)$ if $\chi(G) = r$ and $\theta(G) = s$.

[Epple and Huang (2010)]

**cograph**: a graph not contain $P_4$ (i.e., the path with four vertices) as an induced subgraph.

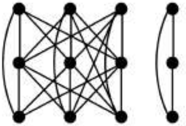
[Corneil, Lerchs and Burlingham (1981)]

**box cograph**: a box cograph is a cograph $G$ having exactly $\chi(G)\theta(G)$ vertices.

The class of box cographs $G$ is denoted by $\mathcal{B}(r,s)$ if $\chi(G) = r$ and $\theta(G) = s$.

[Epple and Huang (2010)]

- **Example:** a box cograph of dimension 3 by 4.



# 2. Known results

**Theorem (Epple and Huang, JGT, 2010)**

*Let G be a graph with* $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, *then* $G \in \mathcal{B}(k, l)$.

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

**Outline of the proof:** Consider a $k$-coloring $S_1, S_2, \ldots, S_k$ and a $l$-clique covering $C_1, C_2, \ldots, C_l$ of $G$.

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

**Outline of the proof:** Consider a $k$-coloring $S_1, S_2, \ldots, S_k$ and a $l$-clique covering $C_1, C_2, \ldots, C_l$ of $G$.

- $G$ is not $(k-1, l-1)$-colorable.
  ($\chi^b(G) > k + l - 2$. $G$ is not $(k', l')$-colorable for some $k' + l' = k + l - 2$. Since $G$ is $(k, 0)$-colorable and $(0, l)$-colorable, $k' \leq k - 1$ and $l' \leq l - 1$. Thus $k' = k - 1$ and $l' = l - 1$.)

## 2. Known results

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

- $G$ is not $(k-1, l-1)$-colorable.

## 2. Known results

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

- $G$ is not $(k-1, l-1)$-colorable.
- $| S_i \cap C_j |= 1$. ($| S_i \cap C_j | \leq 1$. Suppose $| S_i \cap C_j |= 0$ for some $i$ and $j$. Then $\{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k, C_1 \cap S_i, \ldots, C_{j-1} \cap S_i, C_{j+1} \cap S_i, \ldots, C_l \cap S_i\}$ is a $(k-1, l-1)$-coloring of $G$.)

# 2. Known results

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

- $G$ is not $(k-1, l-1)$-colorable.
- $| S_i \cap C_j |= 1$. ($| S_i \cap C_j | \leq 1$. Suppose $| S_i \cap C_j |= 0$ for some $i$ and $j$. Then $\{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k, C_1 \cap S_i, \ldots, C_{j-1} \cap S_i, C_{j+1} \cap S_i, \ldots, C_l \cap S_i\}$ is a $(k-1, l-1)$-coloring of $G$.)
- $G$ has $kl$ vertices.

## 2. Known results

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$, $\theta(G) = l$, $\chi^b(G) = k + l - 1$, then $G \in \mathcal{B}(k, l)$.*

- $G$ is not $(k-1, l-1)$-colorable.
- $\mid S_i \cap C_j \mid = 1$. ($\mid S_i \cap C_j \mid \leq 1$. Suppose $\mid S_i \cap C_j \mid = 0$ for some $i$ and $j$. Then $\{S_1, \ldots, S_{i-1}, S_{i+1}, \ldots, S_k, C_1 \cap S_i, \ldots, C_{j-1} \cap S_i, C_{j+1} \cap S_i, \ldots, C_l \cap S_i\}$ is a $(k-1, l-1)$-coloring of $G$.)
- $G$ has $kl$ vertices.
- $G$ is a cograph.

  (Frequently employ the property: if $G$ is a cograph with at least two vertices then either $G$ or $\overline{G}$ is disconnected.)

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$ and $\theta(G) = l$. Then $\chi^b(G) \leq k + l - 1$, and the following statements $(i), (ii)$ and $(iii)$ are equivalent:*

(i) $\chi^b(G) = k + l - 1$,

(ii) $G$ *is not* $(k - 1, l - 1)$*-colorable,*

(iii) $G \in \mathcal{B}(k, l)$.

**Theorem (Epple and Huang, JGT, 2010)**

*Let $G$ be a graph with $\chi(G) = k$ and $\theta(G) = l$. Then $\chi^b(G) \leq k + l - 1$, and the following statements $(i), (ii)$ and $(iii)$ are equivalent:*

(*i*) $\chi^b(G) = k + l - 1$,

(*ii*) *$G$ is not $(k-1, l-1)$-colorable,*

(*iii*) $G \in \mathcal{B}(k, l)$.

- **An example for the case $\chi^b(G) - \theta(G)$ arbitrarily large.**

  Given positive integers $m$ and $n$, let $mK_n$ denote the disjoint union of $m$ copies of $K_n$. It is clear that $mK_n \in \mathcal{B}(n, m)$, and thus

$$\chi^b(mK_n) = \chi(G) + \theta(G) - 1 = n + m - 1.$$

# 3. Our results

The **clique number** $\omega(G)$: the maximum order over all cliques of $G$.

$$\chi^b(mK_n) = n + m - 1 = \omega(mK_n) + \theta(mK_n) - 1.$$

# 3. Our results

The **clique number** $\omega(G)$: the maximum order over all cliques of $G$.

$$\chi^b(mK_n) = n + m - 1 = \omega(mK_n) + \theta(mK_n) - 1.$$

A natural upper bound: $\chi^b(G) \leq \chi(G) + \theta(G) - 1.$

# 3. Our results

The **clique number** $\omega(G)$: the maximum order over all cliques of $G$.

$$\chi^b(mK_n) = n + m - 1 = \omega(mK_n) + \theta(mK_n) - 1.$$

A natural upper bound: $\chi^b(G) \leq \chi(G) + \theta(G) - 1.$

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and the following statements $(i), (ii), (iii)$ and $(iv)$ are equivalent:*

(i) $\chi^b(G) = \theta(G) + 1$,

(ii) *$G$ is not $(1, \theta(G) - 1)$-colorable,*

(iii) $G \in \mathcal{B}(2, \frac{|V(G)|}{2})$,

(iv) *$G$ is the disjoint union of balanced complete bipartite graphs.*

# 3. Our results

**Theorem**

*Let $G$ be a graph with $\omega(G) < 4$. Then $\chi^b(G) \leq \theta(G) + 2$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + 2$,

(ii) *$G$ is not $(2, \theta(G) - 1)$-colorable,*

(iii) $G \in \mathcal{B}(3, \frac{|V(G)|}{3})$.

**Theorem**

*Let $G$ be a graph with $\omega(G) < 4$. Then $\chi^b(G) \leq \theta(G) + 2$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + 2$,

(ii) *$G$ is not $(2, \theta(G) - 1)$-colorable,*

(iii) $G \in \mathcal{B}(3, \frac{|V(G)|}{3})$.

- Remark: If $\omega(G) < 4$, then $\chi^b(G) \leq \theta(G) + \omega(G) - 1$.

**Theorem**

*Let $G$ be a graph with $\omega(G) < 4$. Then $\chi^b(G) \leq \theta(G) + 2$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + 2$,

(ii) *$G$ is not $(2, \theta(G) - 1)$-colorable,*

(iii) $G \in \mathcal{B}(3, \frac{|V(G)|}{3})$.

- Remark: If $\omega(G) < 4$, then $\chi^b(G) \leq \theta(G) + \omega(G) - 1$.
- Problem: If $\omega(G) \geq 4$ ?

**Theorem**

*Let $G$ be a line graph of a simple graph with $\omega(G) < r+1$, $r \geq 4$. Then $\chi^b(G) \leq \theta(G) + r - 1$, and the following statements (i), (ii) and (iii) are equivalent:*

(i) $\chi^b(G) = \theta(G) + r - 1$,

(ii) $G$ *is not* $(r-1, \theta(G)-1)$*-colorable,*

(iii) $G$ *is the disjoint union of* $K_r$.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

# 4. The proof

**Theorem**

*Let G be a triangle free graph. Then* $\chi^b(G) \leq \theta(G) + 1$*, and* $\chi^b(G) = \theta(G) + 1$ *iff* $G$ *is not* $(1, \theta(G) - 1)$*-colorable.*

**Proof:** Let G be a triangle free graph with $|V(G)| = n \geq 1$ and $\theta(G) = l$.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

**Proof:** Let G be a triangle free graph with $|V(G)| = n \geq 1$ and $\theta(G) = l$.

If $l = 1$, then $G = K_1$ or $K_2$, and thus $\chi^b(G) \leq 2$.

$\chi^b(G) = 2$ iff $G(= K_2)$ is not $(1, 0)$-colorable.

Assume that $l \geq 2$.

# 4. The proof

**Theorem**

*Let G be a triangle free graph. Then* $\chi^b(G) \le \theta(G) + 1$*, and* $\chi^b(G) = \theta(G) + 1$ *iff* $G$ *is not* $(1, \theta(G) - 1)$*-colorable.*

**Proof:** Let G be a triangle free graph with $|V(G)| = n \ge 1$ and $\theta(G) = l$.

If $l = 1$, then $G = K_1$ or $K_2$, and thus $\chi^b(G) \le 2$.

$\chi^b(G) = 2$ iff $G(= K_2)$ is not $(1, 0)$-colorable.

Assume that $l \ge 2$.

Let $C_0 = \emptyset$, and let $C_1, C_2, \ldots, C_l$ be a partition of $V(G)$ s.t. each $C_j$ is a clique.

For $i \in \{0, 1, \ldots, l - 2\}$, let $G_i = G - \cup_{j=0}^{i} C_j$.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

## Theorem

*Let G be a triangle free graph. Then* $\chi^b(G) \leq \theta(G) + 1$, *and* $\chi^b(G) = \theta(G) + 1$ *iff G is not* $(1, \theta(G) - 1)$*-colorable.*

## Lemma (Erdős, Gimbel and Straight, Europ. J. Combin. 1990)

*If G is triangle free graph other than* $K_2$, *then* $\chi(G) = \chi^c(G)$.

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

**Lemma (Erdős, Gimbel and Straight, Europ. J. Combin. 1990)**

*If $G$ is triangle free graph other than $K_2$, then $\chi(G) = \chi^c(G)$.*

Since $(C_{l-1} \cup C_l) \subseteq V(G_i)$, $G_i \neq K_2$. By Lemma,

$$\chi(G_i) = \chi^c(G_i) \leq \theta(G_i) = l - i.$$

Then $V(G_i)$ can be partitioned into $l - i$ independent sets

$$\{S_1, S_2, \ldots, S_{l-i}\}.$$

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G)+1$, and $\chi^b(G)=\theta(G)+1$ iff $G$ is not $(1,\theta(G)-1)$-colorable.*

Then $V(G_i)$ can be partitioned into $l-i$ independent sets

$$\{S_1, S_2, \ldots, S_{l-i}\}.$$

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

Then $V(G_i)$ can be partitioned into $l - i$ independent sets

$$\{S_1, S_2, \ldots, S_{l-i}\}.$$

It follows that $\{S_1, S_2, \ldots, S_{l-i}, C_1, C_2, \ldots, C_i\}$ is an $(l - i, i)$-coloring of $G$.

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

Then $V(G_i)$ can be partitioned into $l - i$ independent sets

$$\{S_1, S_2, \ldots, S_{l-i}\}.$$

It follows that $\{S_1, S_2, \ldots, S_{l-i}, C_1, C_2, \ldots, C_i\}$ is an $(l - i, i)$-coloring of $G$.

i.e., $G$ is $(l - i, i)$-colorable for each $i \in \{0, 1, \ldots, l - 2\}$., and

$G$ is $(l - i + 1, i)$-colorable for each $i \in \{0, 1, \ldots, l - 1\}$.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

If $G$ is $(1, l-1)$-colorable, $\chi^b(G) \leq l$ since $G$ is $(0, l)$-colorable.

If $G$ is not $(1, l-1)$-colorable, $\chi^b(G) = l + 1$ since $G$ is $(1, l)$-colorable and $(0, l+1)$-colorable.

**Theorem**

*Let $G$ be a triangle free graph. Then $\chi^b(G) \leq \theta(G) + 1$, and $\chi^b(G) = \theta(G) + 1$ iff $G$ is not $(1, \theta(G) - 1)$-colorable.*

If $G$ is $(1, l-1)$-colorable, $\chi^b(G) \leq l$ since $G$ is $(0, l)$-colorable.

If $G$ is not $(1, l-1)$-colorable, $\chi^b(G) = l + 1$ since $G$ is $(1, l)$-colorable and $(0, l+1)$-colorable.

It follows that $\chi^b(G) \leq l + 1$, and $\chi^b(G) = l + 1$ iff $G$ is not $(1, l-1)$-colorable.

This completes the proof. □

**Theorem**

*Let $G$ be a triangle free graph on $n$ vertices, $\theta(G) = l$. Then $G$ is not $(1, l-1)$-colorable iff $G \in \mathcal{B}(2, \frac{n}{2})$.*

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph on $n$ vertices, $\theta(G) = l$. Then $G$ is not $(1, l-1)$-colorable iff $G \in \mathcal{B}(2, \frac{n}{2})$.*

**Proof:**

$\Leftarrow$: If $G \in \mathcal{B}(2, \frac{n}{2})$, then $G$ is not $(1, l-1)$-colorable.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph on $n$ vertices, $\theta(G) = l$. Then $G$ is not $(1, l-1)$-colorable iff $G \in \mathcal{B}(2, \frac{n}{2})$.*

**Proof:**

$\Leftarrow$: If $G \in \mathcal{B}(2, \frac{n}{2})$, then $G$ is not $(1, l-1)$-colorable.

$\Rightarrow$: Since $G$ is triangle free, $l \geq \frac{n}{2}$.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph on $n$ vertices, $\theta(G) = l$. Then $G$ is not $(1, l-1)$-colorable iff $G \in \mathcal{B}(2, \frac{n}{2})$.*

**Proof:**

$\Leftarrow$: If $G \in \mathcal{B}(2, \frac{n}{2})$, then $G$ is not $(1, l-1)$-colorable.

$\Rightarrow$: Since $G$ is triangle free, $l \geq \frac{n}{2}$.

Suppose that $G$ is not $(1, l-1)$-colorable.

# 4. The proof

**Theorem**

*Let $G$ be a triangle free graph on $n$ vertices, $\theta(G) = l$. Then $G$ is not $(1, l-1)$-colorable iff $G \in \mathcal{B}(2, \frac{n}{2})$.*

**Proof:**

$\Leftarrow$: If $G \in \mathcal{B}(2, \frac{n}{2})$, then $G$ is not $(1, l-1)$-colorable.

$\Rightarrow$: Since $G$ is triangle free, $l \geq \frac{n}{2}$.

Suppose that $G$ is not $(1, l-1)$-colorable.

Then $l = \frac{n}{2}$, and so $n$ is even.

(If $l > \frac{n}{2}$, then there exists a clique $C_i$ s.t. $|C_i| = 1$, and so $G$ is $(1, l-1)$-colorable.)

**Lemma (Epple, Ph.D. Thesis, 2011)**

*Let $G$ be a nonbipartite triangle free graph on $n$ vertices with $n$ even. Then $G$ is $(1, \frac{n}{2} - 1)$-colorable.*

**Lemma (Epple, Ph.D. Thesis, 2011)**

*Let $G$ be a bipartite graph on $n$ vertices with $n$ even. If $G$ is not a box cograph, then $G$ is $(1, \frac{n}{2} - 1)$-colorable.*

# 4. The proof

**Lemma (Epple, Ph.D. Thesis, 2011)**

*Let $G$ be a nonbipartite triangle free graph on $n$ vertices with $n$ even. Then $G$ is $(1, \frac{n}{2} - 1)$-colorable.*

**Lemma (Epple, Ph.D. Thesis, 2011)**

*Let $G$ be a bipartite graph on $n$ vertices with $n$ even. If $G$ is not a box cograph, then $G$ is $(1, \frac{n}{2} - 1)$-colorable.*

$G$ is not $(1, l-1)$-colorable, $l = \frac{n}{2}$, $n$ is even. By Lemmas, $G$ is bipartite and is a box cograph, i.e., $G \in \mathcal{B}(2, \frac{n}{2})$. □

# 5. Open problems

**Problem**

*Every graph G with* $\omega(G) < 5 \Rightarrow \chi^b(G) \leq \theta(G) + 3$ *?*

# 5. Open problems

**Problem**

*Every graph $G$ with* $\omega(G) < 5 \Rightarrow \chi^b(G) \leq \theta(G) + 3$ *?*

Remark: Every graph $G$ with $\omega(G) < 5 \Rightarrow \chi(G) \leq \theta(G) + 3$ ?

# 5. Open problems

**Problem**

*Every graph G with* $\omega(G) < 5 \Rightarrow \chi^b(G) \leq \theta(G) + 3$ *?*

Remark: Every graph $G$ with $\omega(G) < 5 \Rightarrow \chi(G) \leq \theta(G) + 3$ ?

If this is true, then the problem is solved.

**Proof:** Let $G$ be a graph with $\omega(G) < 5$, $\theta(G) = l$, $l \geq 1$. Let $C_0 = \emptyset$, Let $C_1, C_2, \ldots, C_l$ be a partition of $V(G)$, s.t. each $C_j$ is a clique. For $i \in \{0, 1, \ldots, l-1\}$, let $G_i = G - \cup_{j=0}^{i} C_j$. **If** $\chi(G_i) \leq \theta(G_i) + 3 = l - i + 3$ **is true**. Then $V(G_i)$ can be partitioned into $l - i + 3$ independent sets $\{S_1, S_2, \ldots, S_{l-i+3}\}$. It follows that $\{S_1, S_2, \ldots, S_{l-i+3}, C_1, C_2, \ldots, C_i\}$ is an $(l - i + 3, i)$-coloring of $G$, i.e., $G$ is $(l - i + 3, i)$-colorable for each $i \in \{0, 1, \ldots, l-1\}$. Since $G$ is $(0, l)$-colorable, $G$ is $(t, l + 3 - t)$-color -able for each $t \in \{0, 1, 2, 3\}$. So $\chi^b(G) \leq \theta(G) + 3$.

## 5. Open problems

**Problem (Huang, GTCA(The 8th International Symposium on Graph Theory and Combinatorial Algorithms), 2019)**

*Characterize graphs $G$ for which $\chi^b(G) = \chi^c(G)$.*

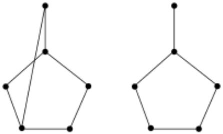
# 5. Open problems

**Problem (Huang, GTCA(The 8th International Symposium on Graph Theory and Combinatorial Algorithms), 2019)**

*Characterize graphs $G$ for which $\chi^b(G) = \chi^c(G)$.*

**Lemma (Ekim and Gimbel, Discrete Math. 2009)**

*The only triangle free graphs with $\chi(G) = \theta(G)$ are $P_3, K_1 \cup K_2, P_4, 2K_2, C_4, C_5$, together with the two graph are depicted below.*



# 5. Open problems

**Problem (Huang, GTCA, 2019)**

*Characterize graphs $G$ for which $\chi^b(G) = \chi^c(G)$.*

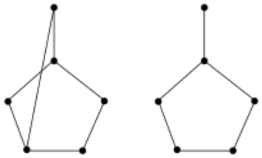
# 5. Open problems

**Problem (Huang, GTCA, 2019)**

*Characterize graphs G for which* $\chi^b(G) = \chi^c(G)$.

**Corollary**

*The only triangle free graphs with* $\chi^b(G) = \chi^c(G)$ *are* $P_3, K_1 \cup K_2, P_4, C_5,$ *together with the two graph are depicted below.*



# 5. Open problems

# 5. Open problems

**Problem (Huang, GTCA, 2019)**

*What can be said about graphs $G$ for which $\chi^b(G) = \chi(G)$ ?*

# 5. Open problems

**Problem (Huang, GTCA, 2019)**

*What can be said about graphs $G$ for which $\chi^b(G) = \chi(G)$ ?*

**Problem (Huang, GTCA, 2019)**

*Characterize planar graphs $G$ for which $\chi^b(G) = \theta(G)$.*

*Thank you!*